Introduction to Earth Gravity

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I. Introduction

Things on the earth's surface – chairs, houses, people - stay on the earth unless subjected to extreme forces. We all stay here due to the gravity of the earth. This has been recognized for thousands of years. The earth also rotates on its polar axis once per day. This has been recognized only for a few hundred years. This rotation also produces a force that tends to throw things outward from the axis.

On of the objections to earth rotation was this outward force. It was ask "Why don't we all fly off the earth?". To some extent we do. But the gravitational attraction of the earth is about 300 times as strong as the rotational outward force, so we stay put. However this force does have it's effects. It causes the earth to change shape, being a little larger at the equator than at the poles. (By about 1/300 of course.)

One effect of this history is the way "gravity" is defined in geodesy. "Gravity" is taken as the observed acceleration. This is the sum of the two effects. Therefore you must be careful when reading technical books and articles mentioning "gravity". Iit is important to know who is writing and which definition of "gravity" is used. Here we will use the geodestist nomenclature:

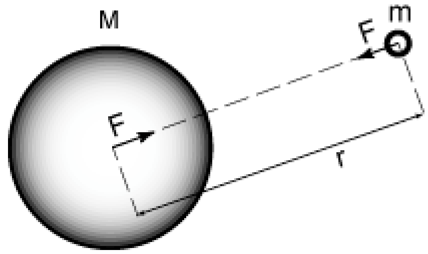
Newtonian Gravitation	The physics book attraction
Rotational Effects	Centrifugal Acceleration
Gravity	The sum of these two.

<u>The earth's gravity field, in geodesy, refers to the sum of the Newtonian and rotation accelerations</u>.

II. Newtonian Gravitational Attraction

All things attract each other due to the force of gravity. This is why planets and stars stay together. It is also why they are (almost) spheres. Below some size this is not true. Then the rigid body forces overcome self gravity. At large enough size, the body is effectively a very thick fluid and flows to the lowest energy shape. This is a spherical, if no other forces are involved. However, most planets and stars rotate, this brings in rotational forces and changes the shape.

The figure below illustrates Newtonian gravitation. For spherical bodies, the force acts as if all the mass is at the center of the body.



$$\left|\vec{F}\right| = \frac{GMm}{r^2}$$

Newton's Law of Gravity

For a small mass, m, on earth's surface the distance is the radius, r_e, and

$$\boxed{\left|\vec{F}\right| = mg \quad \text{with} \quad g = \frac{GM}{r_e^2}}$$

For the earth, the mass, M, is constant. The value “m” is the mass of the test object and the separation, r, is about the earth radius. “G” represents the universal gravitational constant. (This is the least well know universal physical constant. For astronomical objects – stars, planets, moons - we know the product GM much more accurately than we know either factor.)

For the force of gravity on the earth we can group values in the force equation to obtain an equation for the acceleration of gravity, g. These factors have the units of acceleration. Given a value of G and a measurement of g and r, the mass of the earth can be determined.

In this simple model, the only reason that the acceleration of gravity would vary is changes in altitude, that is changes in the radius. However the real earth is more complicated, mainly because it rotates once an day.

III. Gravitational Potential Energy

If you have an object – say a baseball – of mass m on a table it has some potential energy. Let it roll off the table and it falls, converting the potential energy into kinetic energy. When it stops bouncing on the floor, it will have a new, lower potential energy. This is the energy you must expend to bring it back to the table top. If the height is h, then the change in potential energy, ΔV , will be

$$\Delta V = mgh\,.$$

This is the force, mg, times the distance moved h. Work is force times distance.

Iit is more fundamental to think of the potential, V, as existing at each point in space. The force is then the negative of the change in potential over a unit distance. That is the force is the negative of the rate of change of the potential energy. More mathematically the force is the derivative[1] of the potential with respect to distance, s,

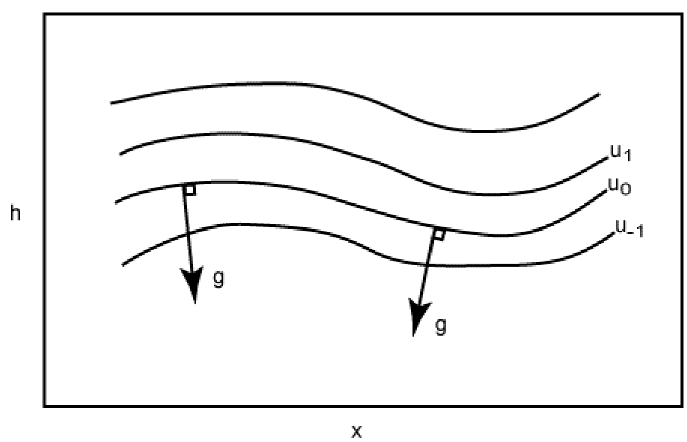
$$F = -\frac{dV}{ds}\,.$$

(Note here the up direction is defined as positive. A negative force means a downward force. However, many places in geodesy use the convention that down is positive. In these cases the negative sign of often left off the relation between force and potential.)

In three dimensions this is slightly more complex, being a vector quantity. This is written as the “gradient”

$$F = -\vec{\nabla} V$$

This complexity can be simplified by plotting out lines or surfaces of constant potential. Then the gradient is just the rate of change in the direction where it is greatest in magnitude. This vector is perpendicular to the surface there. This gives the direction of the force.

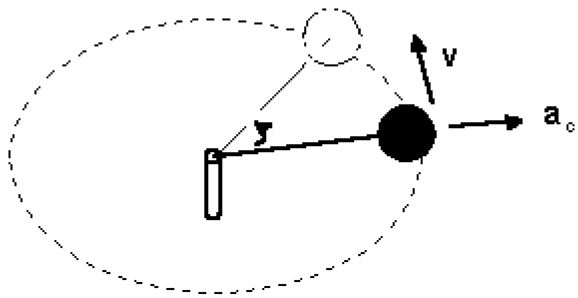


The potential in the above figure is called U, which is the total potential, including the rotational effects.

[1] There is a separate technical note giving an overview of the derivative and integral.

IV. Rotational Effects

If you put a small weight on a string and swing it in a circle, there is a force that tries to pull the weight outward.

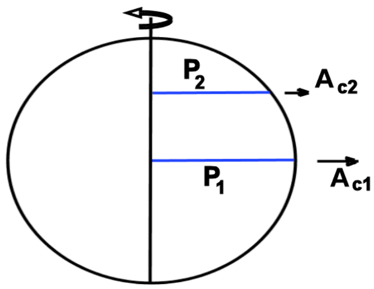


Centrifugal Acceleration

Radius	r		
Period	T		
Frequency	f	$= 1 / T$	
Angular Frequency	ω	$= 2\pi f$	$= 2\pi / T$
Velocity	v	$= 2\pi r / T$	$= \omega r$
Centrifugal Acceration	a_c	$= v^2 / r$	$= \omega^2 r$

The equations for the period, frequency and outward acceleration are given above. Notice that this acceleration is a vector. It's direction is outward, that is from the center of the rotation along the string. The magnitude is $\omega^2 r$.

This acceleration acts on everything that rotates with the earth. In this case the period is 1 day (really 1 sidereal day – 3 min 56 sec shorter than 24 hours) and is the same everywhere on the earth. The moment arm, called "r" above, is not the distance from the center of the earth, but the distance from the spin axis to the surface. It will be a function of latitude. While the period is the same everywhere on the earth, the moment arm is now. This makes the centrifugal acceleration a function of latitude.



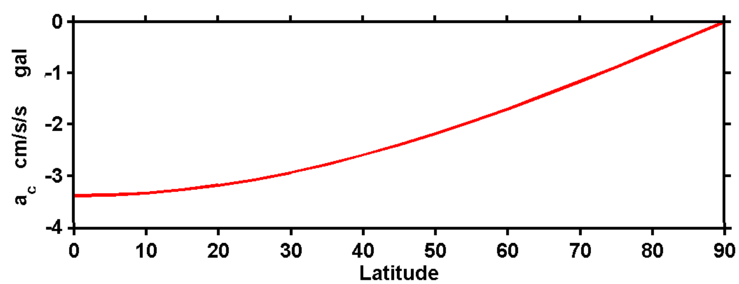
Rotation Effect
Create Equatorial Bulge

This moment arm, often called "p", will be maximum at the equator and zero at the poles. Therefore the centrifugal acceleration is maximum at the equator and zero at the poles. This centrifugal acceleration will be directed along the line that goes from the spin axis to the surface. This line is not perpendicular to the surface.

On the equator the moment arm is the radius of the earth there, about 6378 km. This gives a rotational acceleration of

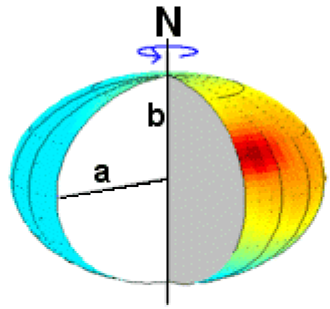
$$a_c = \left(\frac{2\pi}{86164\,\mathrm{sec}}\right)^2 \times 6378000\, m = 0.0339\, m/s^2 = 3.39\, cm/s^2$$

The total acceleration of gravity is about 9.80 m/s^2, so the maximum rotational acceleration is about 1/300 of the total. The rotational force is zero at the poles because the moment arm is zero there.



This figure gives the centrifugal acceleration as a function of latitude. The positive down convention is used here. The units are in cm/s^2 . Most geodesy uses mks (meter-kilogram-sec) units, except for accelerations of gravity. The cgs system unit even has a special name, the Galileo. It is abbreviated as "gal". One gal is one cm/s^2. In earth gravity work, the most common unit is the milligal, called a mgal. This is about a millionth of the earth's mean value.

In response to the centrifugal force, the earth, which acts almost as a fluid over long time frames, bulges out at the equator. The shape of the earth is no longer a sphere, but that of an ellipse rotated about the spin axis.



Because these two accelerations are vectors, they must be added as vectors. They pull in different directions. However, the deformation of the earth happens in just the right amount that the sum of these two acceleration is always perpendicular to the ellipse of revolution. (Called an ellipsoid or sometimes a spheroid.)

The ellipsoid is defined by two parameters. These can be the longer equatorial radius, often called "a", and the shorter polar axis, called "b". For the earth a is about 6378 km and b about 6356 km. The pole is 23 km nearer the center of the earth than the equator.

V. Earth Gravitational Potential Energy

The total force of gravity can be expressed as the spatial rate of change of a potential

$$U = V + \frac{1}{2}\omega^2 p^2$$

where V is the Newtonian Gravitational Potential discussed above, and the second term is will give rise to the centrifugal force outward from the polar axis.

U acts like the potential energy. The value of U changes by mgh when you move the baseball from the floor to the table h meters higher. Water, and fluids will flow to the lowest value of U. Along a surface of constant U fluids will not flow due to gravity forces. The force of "gravity" is perpendicular to surfaces of constant U. Surfaces of constant U are called <u>level surfaces</u>.

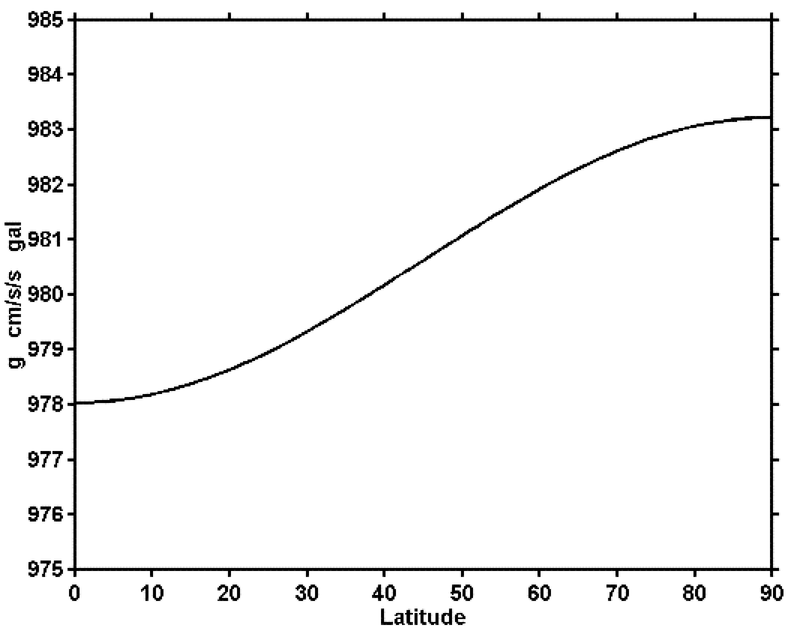
Because the earth, over long time frames, acts like a fluid,

1. <u>the ellipsoid is a level surface – a surface of constant U,</u>
2. <u>the force of gravity, g, is perpendicular to the ellipsoid</u>.

The ellipsoid is a level surface, a surface of constant U. It is not a surface of constant gravity acceleration, g. The acceleration is generated by the gradient in U. That is how closely spaced are surfaces of evenly spaced values of U. Where these surfaces are closer together, gravity is stronger.

For the ellipsoidal model of the earth, the surface gravity is only a function of latitude. The gravity acceleration is stronger at the poles because the earth's surface is closer to the center of the earth and the centrifugal acceleration is smaller. This was first discovered by Newton from the changes in the period of good clocks calibrated in Paris and used near the equator. (The difference was about 2 minutes per day.)

The total gravity acceleration values are shown in the following figure.

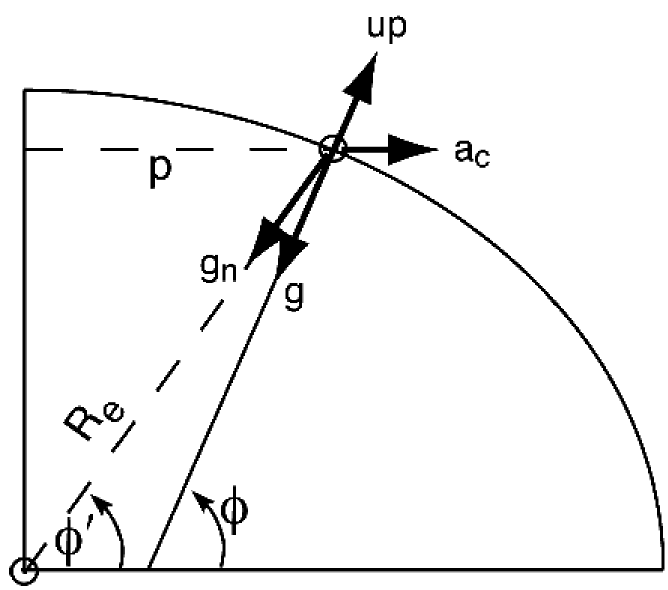


Total Gravity Acceleration

This curve is the value of the acceleration of gravity due to the model ellipsoid. It is called the "nominal" gravity acceleration and often denoted by γ. Real world values in geodesy are reported as differences from this nominal value.

VI. Gravity Effects on Definition of Latitude

The "down" direction is perpendicular to the ellipsoid. This line does not point to the center of the earth, except on the equator and at the poles.

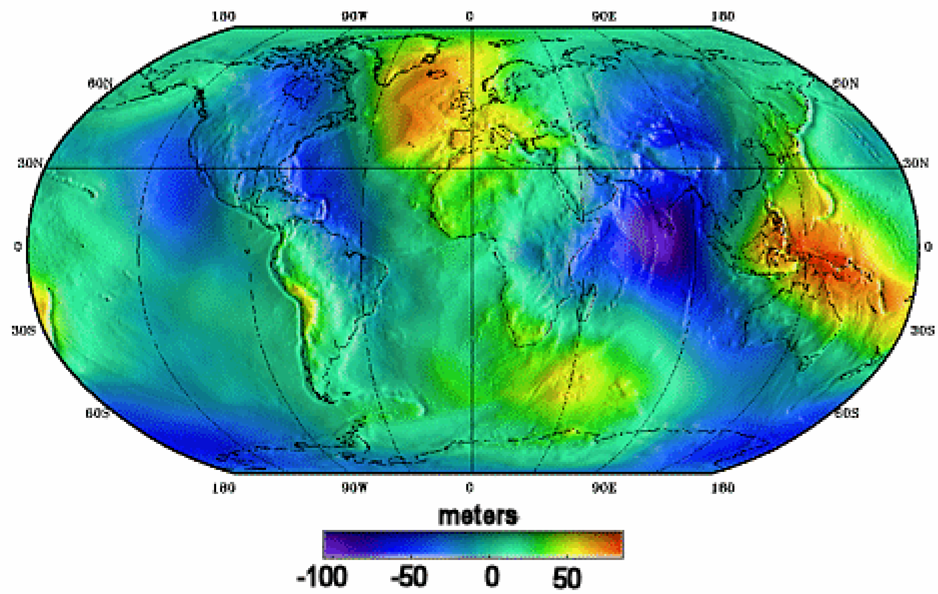


The value of latitude we use on maps is defined not by the line to the center of the earth but by the local value of vertical – that is the perpendicular to the ellipsoid. This is called geodetic latitude, or just latitude. It is usually given the symbol ϕ. The latitude defined by line to the center of the earth is called geocentric latitude and often denoted ϕ'. The geocentric latitude is commonly used only in work with earth satellites.

VII. Real World Gravity – The Geoid

The ellipsoidal model of the earth does a very good job of defining the earth. It is used as a reference for variations that describe the real earth. A very important surface is the geoid. Due to mountains, valleys, ocean basins etc, the level surfaces (surfaces of constant U) are bumpy. The variations in density under the earth's surface also has a major effect. The level surface that is at the mean potential of an ideal ellipsoidal earth is called the geoid.

The height of the geoid above the model ellipsoid is measured and plotted to define the geoid. The distance from the ellipsoid to the geoid is called the geoid separation, the geoid undulation or just the undulation. It is usually denoted by "N". A model of the geoid is shown below.



Geoid Undulations
Earth Gravity Model 96

The geoid goes above and below the reference ellipsoid. In fact one common way to choose the value of U to use is to require that the average of the geoid separations over the earth is zero. The geoid separations vary from about –100 m to plus 100 m.

A very crude approximation for the change in gravity acceleration from the value of the reference ellipsoid (see section V) is

$$\Delta g \approx \frac{-2Ng}{r}$$

where g is the nominal value of the acceleration, r is the radius of the earth and N is the separation. For a separation of 100 m this implies a variation in gravity of 30 x 10^{-3} cm/s^2 or 30 mgal. This is the correct order of magnitude, but real world values are often 2 to 4 times this.