

Radius of the Earth - Radii Used in Geodesy

James R. Clynych

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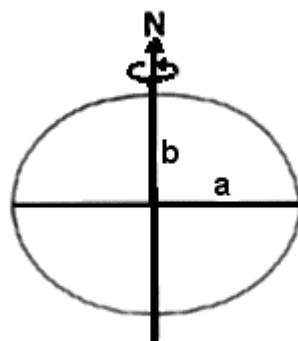
I. Earth Radii Uses

There is only one radius of a sphere. The earth is approximately a sphere and therefore, for some cases, this approximation is adequate. The single radius is used for two main purposes:

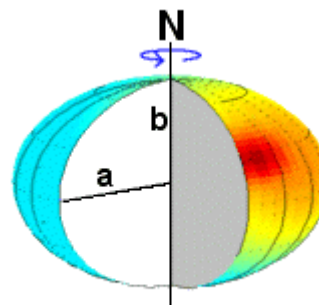
1. To find the distance between the surface and the center of there sphere. (In addition to giving the size of the earth, this value can be subtracted from the local real radius to get the "height" of the point.)
2. To determine the distance on the surface between two points given the end point latitudes and longitudes.

The first use calls for the physical radius of the earth. The second use call for the "radius of curvature" of the earth. For a sphere, these are identical.

In reality the earth is "flattened" at the poles due to the rotation of the earth. The polar radius is 23 km (14 miles) shorter than the equatorial radius. A better approximation is that of an ellipse that is rotated about the polar axis to form a solid.



Rotate Ellipse



Making Ellipsoid

This model is accurate to about 100 m (300 ft).¹ In this case at each point on the earth there are three independent "radii" that are used. The distortion is a small fraction of the total, about 1/300. In general you can use a spherical model if computations at the 1% level are adequate. (Some errors have a maximum of 0.33 %, some twice this.)

¹ This does not mean the earth's surface is modeled at the 100 m level by an ellipsoid. The geoid, which is essentially mean sea level, differs from an ellipsoid by less than 100 m. (Earth Gravity Model 1996). If the earth were homogeneous the fit would be exact.

II. The Three Radii of Earth and Their Use

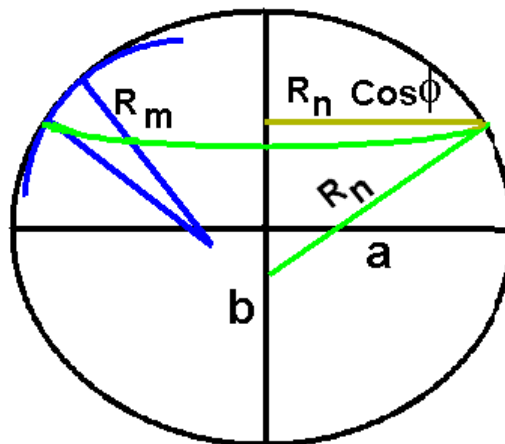
There are three radii that come into use in geodesy. Each is a function of latitude in the ellipsoidal model of the earth. The physical radius, the distance from the center of the earth to the ellipsoid is the least used. The other two are called the "radii of curvature".

If you wish to convert a small difference of latitude or longitude into the linear distance on the surface of the earth, then the spherical earth equation would be,

$$\begin{aligned}dN &= R d\phi, \\dE &= R \cos\phi d\lambda\end{aligned}$$

where R is the radius of the sphere and the angle differences are in radians. The values dN and dE are distances in the surface of the sphere. They correspond to the small changes in angles, $d\phi$ and $d\lambda$ expressed in radians. For an ellipsoidal earth there is a different radius for each of these directions.

The radius used for the longitude is called the "Radius of Curvature in the prime vertical". It is denoted by R_N , N , or ν , and a few other symbols. (Note that in many geodesy works the symbol N is used for the quantity called the geoid undulation.) This radius of curvature, R_N , has a nice physical interpretation. At the latitude chosen go down the line perpendicular to the ellipsoid surface until you intersect the polar axis. In general, this line will not end at the center of the earth. Perpendicular is also called "normal" in mathematics. Thus the subscript "N" is used for this curvature, R_N .



Radii of Curvature

The radius used for the latitude change to North distance is called the "Radius of Curvature in the meridian." It is denoted by R_M , or M , or several other symbols. The subscript "M" comes from meridian, the name of the lines that run north-south on a globe. R_M is the radius of a circle that is tangent to the ellipsoid at the latitude and has the same curvature as the ellipsoid in the north-south direction there. (This circle "kisses" the ellipsoid.) R_M has no good physical interpretation as the termination point changes with latitude.

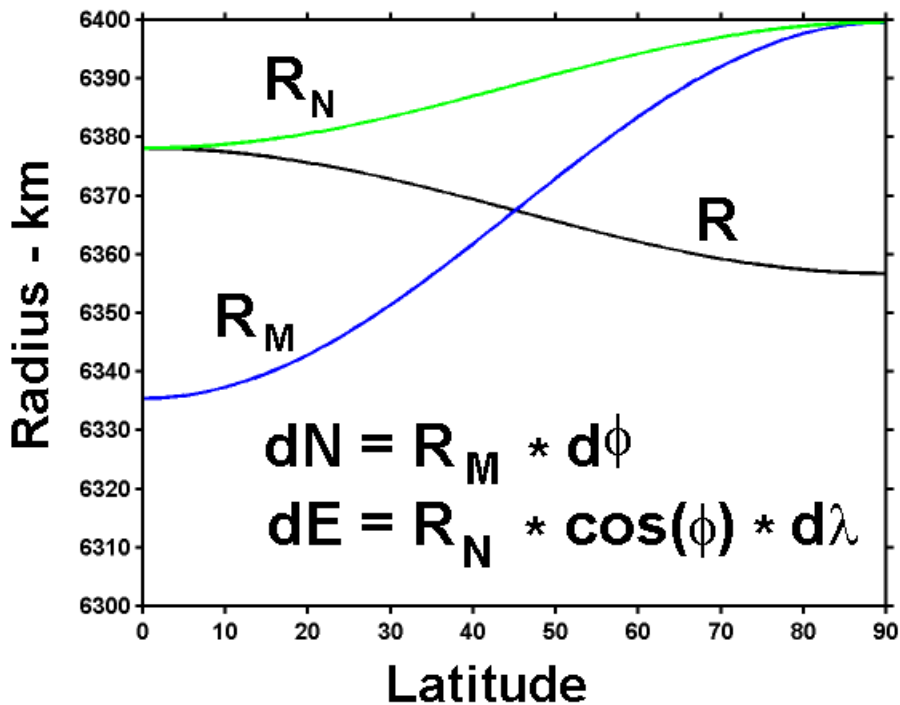
The equations for the relation between the differential distances and angles for an ellipsoid now use two different radii:

$$dN = R_M d\phi,$$

$$dE = R_N \cos \phi d\lambda$$

The radii of curvature R_M and R_N are used in place of the single radius of the sphere.

For the earth these three radii are shown in the following diagram. (To be precise, these are the WGS 84 ellipsoid values.)



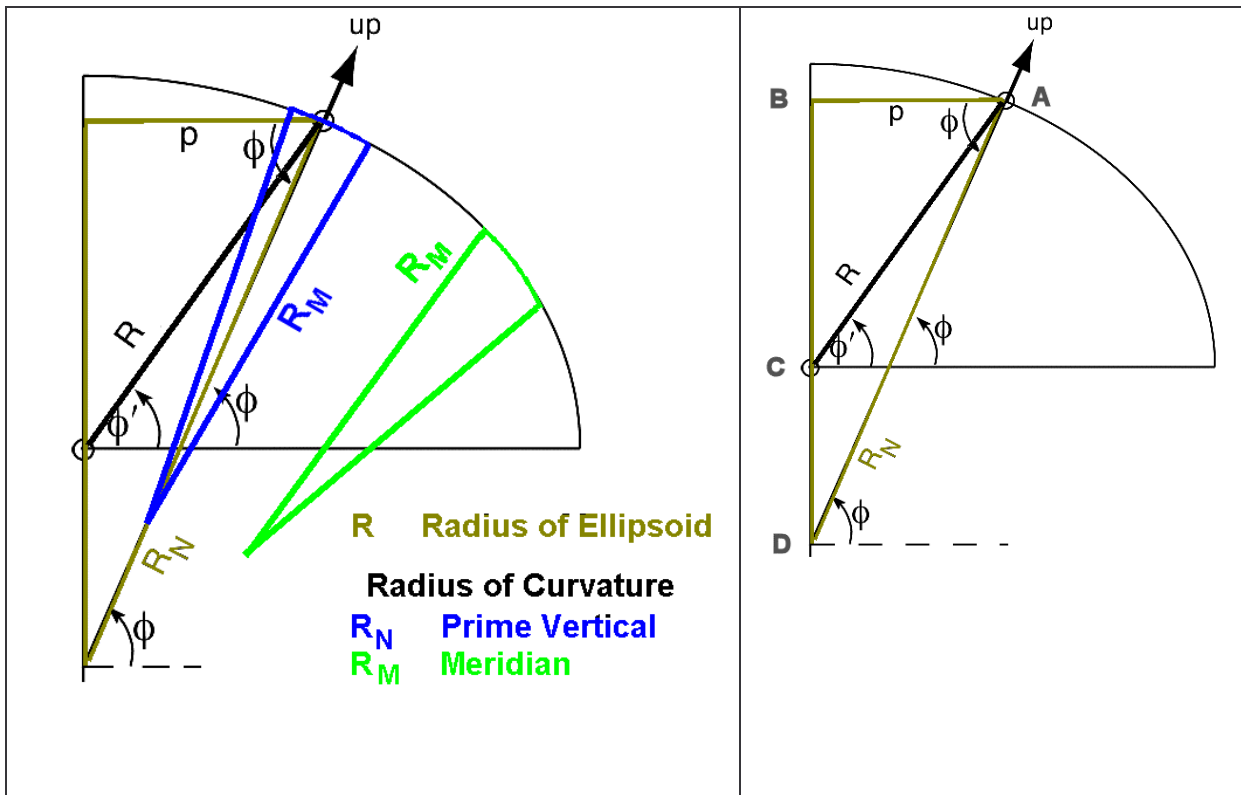
**Earth Radius
Radii of Curvature vs Latitude**

Notice that the both radii of curvature are largest at the poles where the earth is flattest. The radius of curvature for a straight line is infinite. Therefore the flatter the surface, the larger the radius of curvature. The two radii of curvature are the same at the poles. The physical radius is maximum at the equator and minimum at the poles.

III. Diagrams of Radii of Earth

A diagram of these radii is shown below. Note that the geodetic latitude (ϕ , or ϕ_g), is used in geodesy. It is the latitude used on maps. This should not be confused with the geocentric latitude (ϕ' , or ϕ_c) also shown on the figure.

The radius of curvature in the meridian, R_M , is shown for two different latitudes. Two lines and an arc of the circle tangent to the ellipse are shown to illustrate the origin of this radius.



The important auxiliary line, p , is included. This is the moment arm length for the rotational accelerations. In the right diagram, consider the two triangles ABC and ABD that both include the moment arm p . Two equations for p can be written, one using the geodetic latitude and one the geocentric latitude.

$$\begin{aligned}
 p &= R_N \cos \phi \\
 &= R \cos \phi_c
 \end{aligned}$$

This is one of several equations relating the two latitude types.

IV. Equations for Radii of Earth

The equations for these radii are given in the following table.

Radius	Formula
Radius of Curvature: in Prime Vertical, terminated by minor axis	$R_N = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}}$
Radius of Curvature: in Meridian	$R_M = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}$ $= R_N \frac{1 - e^2}{1 - e^2 \sin^2 \phi}$
Radius of ellipsoid	$R = \frac{a \sqrt{[(1 - e^2)^2 \sin^2 \phi + \cos^2 \phi]}}{\sqrt{1 - e^2 \sin^2 \phi}}$
Radius of Curvature: At azimuth α	$\frac{1}{R_\alpha} = \frac{\cos^2 \alpha}{R_M} + \frac{\sin^2 \alpha}{R_N}$

A note of caution on the use of these equations is in order. The quantity ϕ here is the geodetic latitude. Ellipses occur in both geodesy and satellite work. Earth satellite work involves both. But the equations are different in the two fields, while the symbols may be the same, but have different meanings. In geodesy the geodetic latitude, ϕ , is used. This quantity is rarely used in satellite work, where the geocentric latitude is common. This uses the line from the surface to the center of the earth. It may use the same symbol as used here for geodetic latitude.

The formula for the radius of curvature at arbitrary azimuth points up that the fact that the fundamental mathematical quantity is the inverse of these radii, which are simply called curvatures.

The value of the eccentricity, e , used in these equations is given by

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

with "a" being the semi-major axis (equatorial radius) and "b" the semi-minor axis (polar radius) of the ellipsoid. Notice that

$$1 - e^2 = \frac{b^2}{a^2} .$$

The term $1 - e^2$ is common in ellipse equations. It just is the ratio of the axes squared.

The flattening, f , is defined as

$$f = 1 - \frac{b}{a} .$$

The flattening of the earth is about 1/298.25. Notice that $1 - f = b/a$.

It is common in geodesy to define the size and shape of an ellipsoid by giving a and f . However the eccentricity occurs more frequently in equations. The flattening is related to the eccentricity by

$$e^2 = 2f - f^2$$

V. Other Useful Equations Involving Radii of Earth

The area of an ellipse with semi-major axis of a and semi-minor axis of b is given by:

$$A = \pi a b .$$

The volume of an ellipse of revolution, revolved about the semi-minor axis, is given by,

$$V = \frac{4}{3} \pi a^2 b .$$

If you want a sphere of the same volume as an ellipse of revolution you need a radius R_S given by,

$$R_S^3 = a^2 b .$$

The average radius of curvature at any latitude is the geometric mean of R_N and R_M . That is:

$$\begin{aligned} \bar{R}_c &= \frac{1}{2\pi} \int_0^{2\pi} R_\alpha d\alpha , \\ &= \sqrt{R_N R_M} \end{aligned}$$

where α is the azimuth.