Derivative and Integral:
Some Concepts for Geodesy

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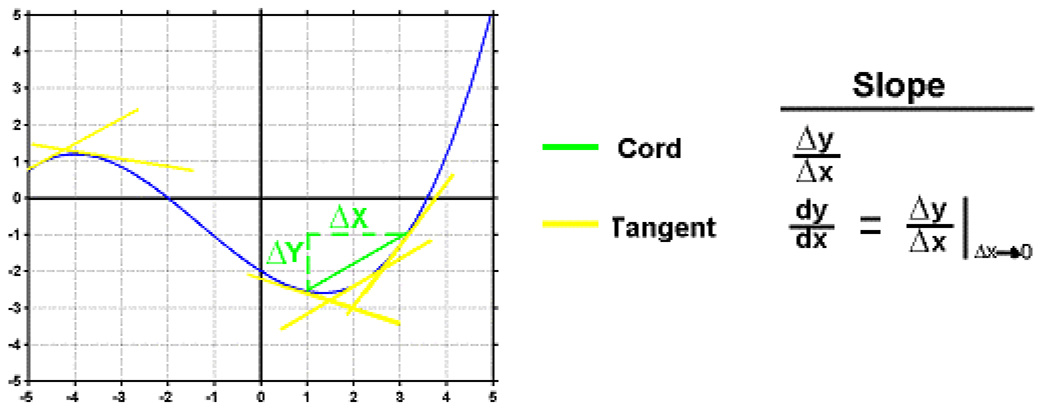
I. Rates of Change (Derivatives) and Integrals

There are many places in physics where the rates of change occur. This is the derivative of calculus. It is not hard to understand. Measure the speed of a car by placing two sensor cables across the road as was done before the invention of the radar gun. The time difference in the pulses as the car crosses the cables is the separation, s, divided by the velocity averaged over the separation of the two sensor cables,

$$\Delta t = \frac{s}{v_{avg}} ,$$

so the average velocity is $s/\Delta t$. This is an average velocity. As the cables come closer, the measurement becomes closer to the instantaneous velocity of the car. This limit is the derivative.

Plot the graph of a function as shown below. The cord, the line between two points, has a slope that is the like the average velocity. As the length between the contact points of the cord get shorter, the cord becomes the tangent line to the curve. There will be a separate tangent line at each point. The slope of the tangent line is the derivative of the curve. This exists at each point.



Derivative of a Curve

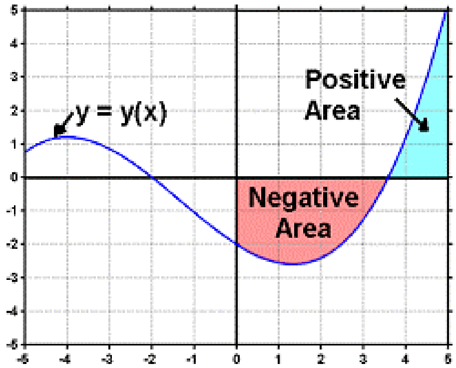
Often we need to do the inverse of taking a derivative. This is done with the integral of the curve. The fundamental theory of calculus says that "the integral is the inverse operation of the derivative".

$$f(x) = \int \frac{df}{dx}\, dx$$

The integral is the area "under" the curve. In mathematical language, the integral is the limit of the areas of a series of boxes of width Δx and with height being f(x) at the midpoint.

$$S = \sum_i f(x_i)\, \Delta x$$

The integral is the limit of this sum where the width of the boxes goes to zero and the number of boxes goes to infinity. The area under the curve is thus the signed sum of the area, where the area below the axis is negative and that above is positive.



$$Y = \int_0^5 y(x)\,dx$$

$$= \text{Area "Under" Curve}$$

Integral — Area Under Curve

There are two different types of integrals. The integral above is called a definite integral. It is between two specific x values. You can also take the integral using a variable for the upper limit. In this case you get a function, not a number.

The further details of the calculus can be found in many textbooks. Derivatives are usually easy to compute. There are a series of runs that can be applied. The same is true of integral, but they can be harder to find in a table. You sometimes have to manipulate them to get them into a standard form.

II The Gradient – Vector Derivative [Advanced Topic]

In physics, and in particular in considering the gravity field of the earth, a multi-dimensional version of the derivative is used. This is called the gradient operator and is usually written as $\vec{\nabla}$. It acts like a vector. A brief overview of this operator is given here along with the few important properties for the geodesy application. For more details, look at mathematics, physics or engineering text books.

Mathematically the gradient can be described in several ways. One of the easiest is in terms of Cartesian components. (See the technical note on vectors for details on the Cartesian representation of a vector.)

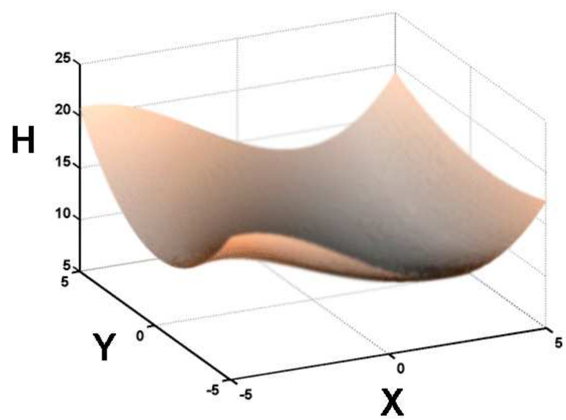
If a function depends on two or more variables, such as the position values of latitude and longitude, then you can take the derivative with respect to each of these independent variables assuming that the other variable is constant. This is called the partial derivative. If f = f(x, y) then the partial derivative with respect to x is written as

$$\frac{\partial f(x, y)}{\partial x} = \frac{\partial f}{\partial x},$$

and the partial with respect to the second independent variable, y, as

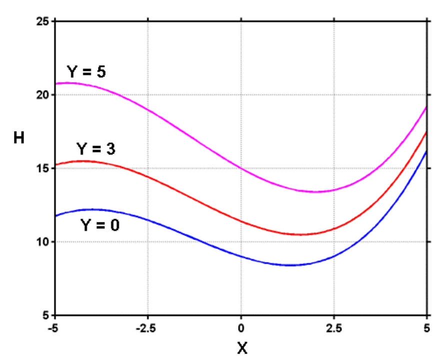
$$\frac{\partial f}{\partial y}.$$

Consider the following diagram of the height in some gentle rolling prairie.

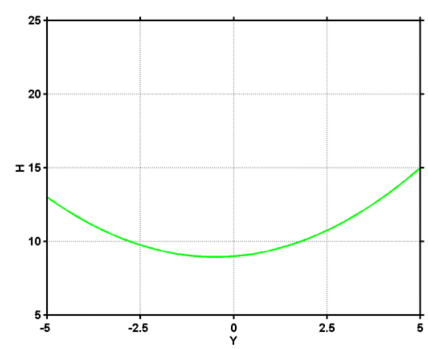


Height In Two Dimensions

The height is our function f. It depends on two horizontal values called X and Y here. For these fixed values we can plot graphs in 1 dimension.

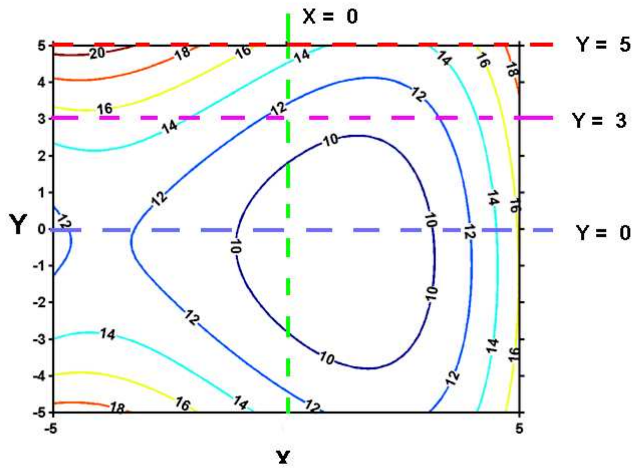


Height Cuts Through Constant Y Planes



Height Cut Through Constant X Plane

Below is a contour plot of the height.   This shows lines of constant height or level. They are also levels of equal gravitational potential energy. The cuts for the four lines plotted are marked with dotted lines.



Height Contour Plot
Showing Lines of Cut Plots

The derivatives (slope of the tangent lines) of the height along the cuts are partial derivatives of the two dimensional figure. There is a derivative at each value of X in the left figure. Notice that the Y=0 cut is only a function of X. Now we could have just as easily taken the Y cut at a value of Y=3 or Y=5, which are the also shown. . Those curves are different, the derivative (partial derivative of the 2-dimensional function) is different from the Y=0 cut. So a partial derivative is labeled with the values of both the independent variables. The partials are functions over the same domain as the original function. And there are two of them in the two dimensional case. For the three dimension case there are three partial derivatives.

These two different partials are combined to form a vector, the gradient. The gradient of f, is given by:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y$$
$$= \left(\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}\right)$$

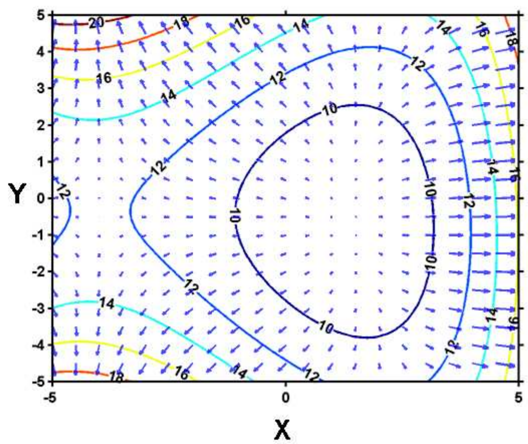
where $\hat{e}_x$ is the unit vector in the x direction and $\hat{e}_y$ is the unit vector in the y direction. The second line is just another notation for the same vector. We see that the gradient operator takes a scalar function and produces a vector.

The meaning of the gradient is simple: it is a vector giving the slope of the two dimensional (or three dimensional) surface. Here in two dimensions it is fairly easy to interpret. It points in the direction of the maximum increase of the function. The length of the vector is equal to the slope in that direction.

Now below is the same contour plot, with small vectors added for the gradient. At each point there is a gradient. We see several things.

1. The magnitude of the gradient is related to the slope of the surface. Where the slope is large, the contour lines are close together. There the gradient is largest.

2. Note that the gradients are always perpendicular to the contour lines. This is a key feature for geodesy.



Height Contour Plot
Showing Gradient Vectors

The gradient point uphill in the direction of maximum slope. This is a gradient of height, but that is also a gradient of gravitational potential energy. Because the force is given by

$$\vec{F} = -\vec{\nabla} V$$

with a negative sign, the force is downhill. Where the gradient is high, it takes more energy to climb the hill. Or water will flow downhill faster. Where the gradient is zero there is no force in the X-Y plane.