Equations of an Ellipse

James R. Clynch
Clynchg3c.com, Austin TX

Newell Garfield
San Francisco State University, San Francisco, CA

August 2006

I. Introduction and Notations

The equations of an ellipse are well known. However, the ellipse is used in several different fields, mathematics, astronomy, earth satellites, geodesy etc. and these fields have different notations, use different origins for coordinates, and different angles. This document will attempt to list the major conventions and the common equations of an ellipse in these conventions.

An ellipse is a two dimensional closed curve that satisfies the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The curve is described by two lengths, a and b. The longer axis, a, is called the semi-major axis and the shorter, b, is called the semi-minor axis. The parameters of an ellipse are also often given as the semi-major axis, a, and the eccentricity, e,

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

or a and the flattening, f,

$$f = 1 - \frac{b}{a}.$$

In the above common equation two assumptions have been made. First that the origin of the x-y coordinates is at the center of the ellipse. Second that the longer axis of the ellipse is along the x-axis.

The convention that the semi-major axis is the x-axis will be used throughout. In this technical note both conventions for the coordinate system origin will be used. The equations with the origin at the center of the ellipse and at one focus are shown.

II. Coordinate Origins and Angles

The first major difference in ellipse notation is the location of the origin. In geodesy and earth sciences the center of the ellipse is used as the origin. In astronomy and earth satellites work, one focus of the ellipse is taken as the origin. In mathematics both cases occur. The two focus points are located along the x-axis (the longer axis) at a distance of

$$c = \sqrt{a^2 - b^2}$$

from the center. There are two, one on each side (at $\pm c$).

The second major difference is what angle is used to describe the position along the ellipse. This note will use the common convention that the angle is measured counter clockwise from the x-axis, that is from the semi-major axis direction. In the case of focus centered origin, the common convention is that the zero of this angle occurs when the radius (from the focus) is minimum. (That is at perigee for earth satellites and perihelion for planets.)

In astronomy the angles of various sorts are often called “anomalies”. So the names of “True Anomaly”, “Eccentric Anomaly”, and “Mean Anomaly” show up in satellite and astronomic work.

Below is a diagram of an ellipse and the two common coordinate systems. In this document the convention is used that upper case letters and symbols are used for the focus centered coordinates (in blue in Figure 1) and lower case for the ellipse center origin case (in green below).

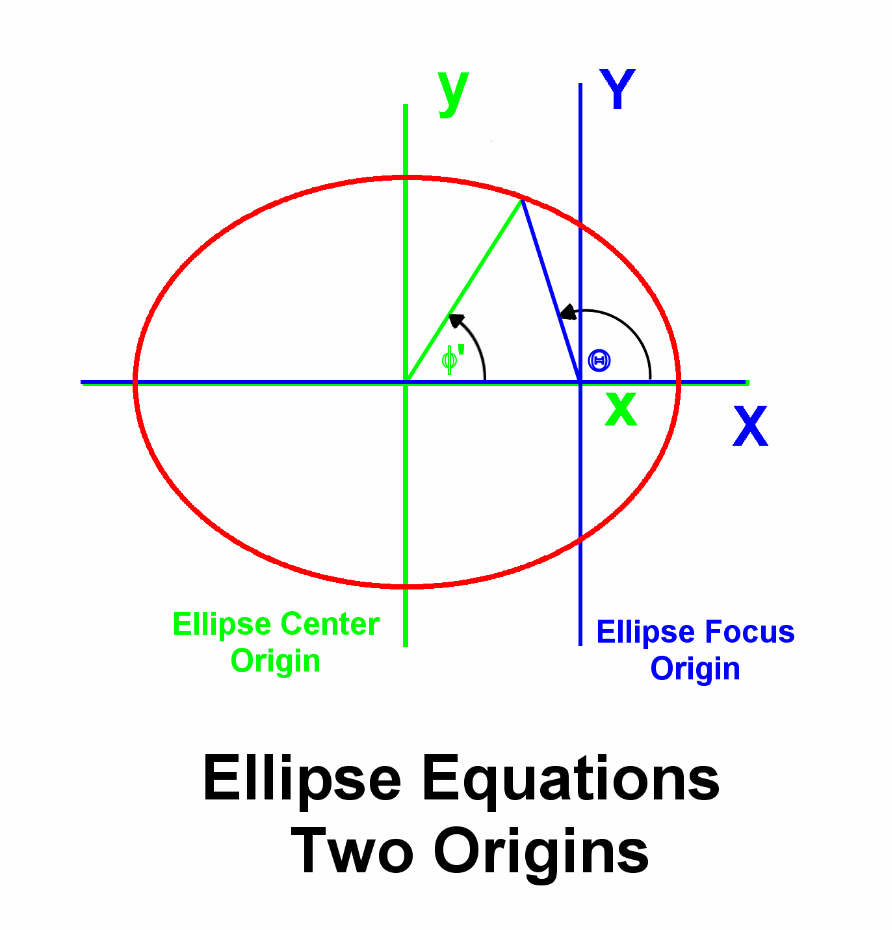
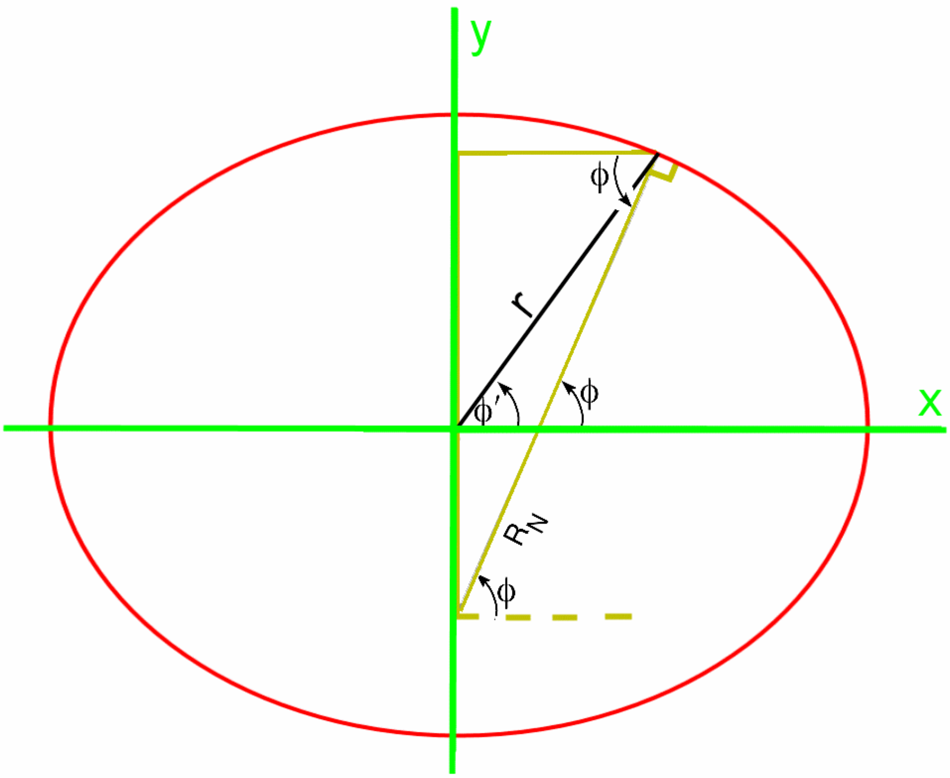


Figure 1

In this figure two angles are shown. The angle from the center to a point on the ellipse is called the geocentric latitude in earth work. (It is not the latitude on maps!). The angle from the focus is called the True Anomaly. This work will not describe how to find the position of a satellite along the elliptical path it takes. This is done with Kepler's equation. See works on astronomy or earth satellites for details.

The earth is an ellipse revolved around the polar axis to a high degree of accuracy. Therefore the equations of an ellipse come into the computation of precise positions and distance on the earth. In the x-y axis convention used here, the situation is shown in Figure 2. In geodesy the axis labeled "y" here is the polar axis, z.



Center Coordinates
Two Latitudes and Two Radii

Figure 2

In Figure 2 the two common latitudes are shown. Geocentric latitude is measured from the center of the earth. It is usually denoted by ϕ'. The latitude shown on maps is called geodetic latitude. It is measured using the perpendicular line to the ellipse (which is also the local down as measured with a plumb bob – to within some small corrections). This line intersects the x-axis at the angle of the geodetic latitude denoted as ϕ. This is also the angle made with the line from the ellipse to the rotational axis.

Two radii are shown in Figure 2. The radius of the ellipse from the center is denoted r. Extending the "down line" to the rotational axis, the distance is denoted by $R_{N.}$ This is the radius of curvature associated with longitude differences (east-west). R_N occurs in many geodetic equations, so it is shown here. (There is also a radius of curvature associated with latitude differences - north-south – motion, but it has no simple physical interpretation. See the Technical Note on "Radius of the Earth" for details.)

There is another angle often used in ellipse equations, both in astronomy and earth sciences. This is shown in Figure 3.

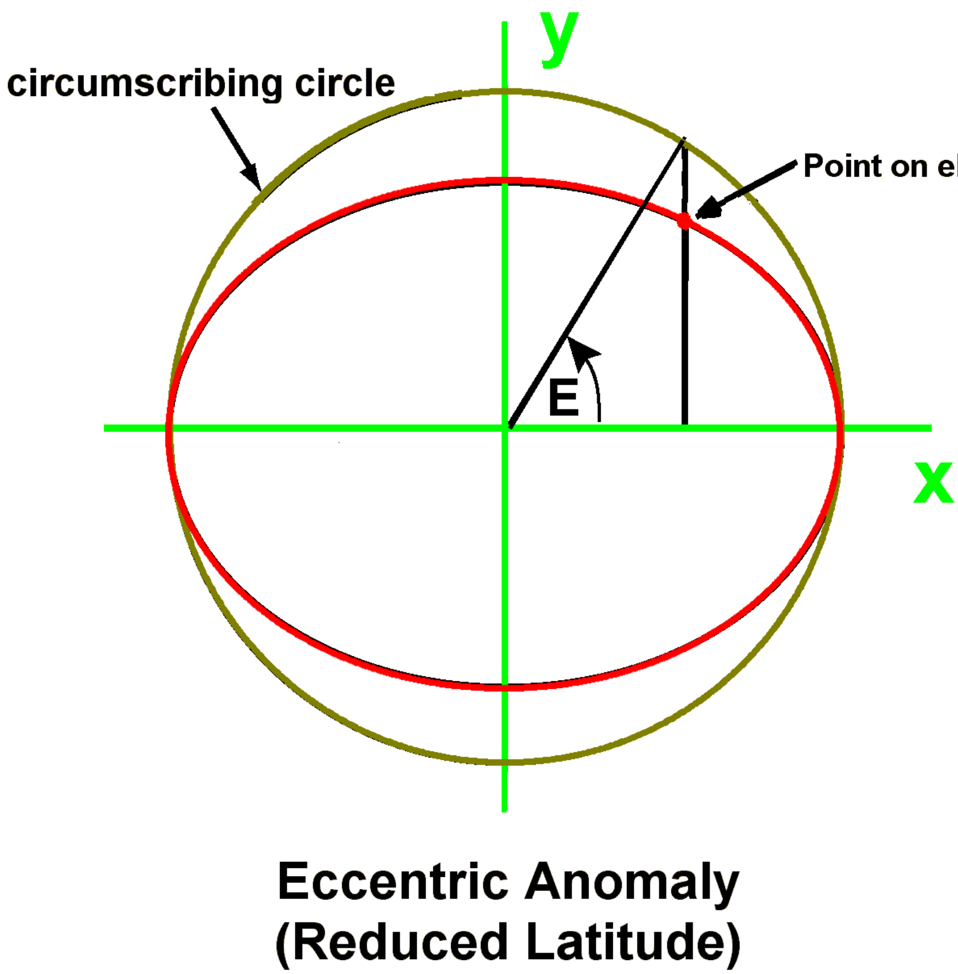
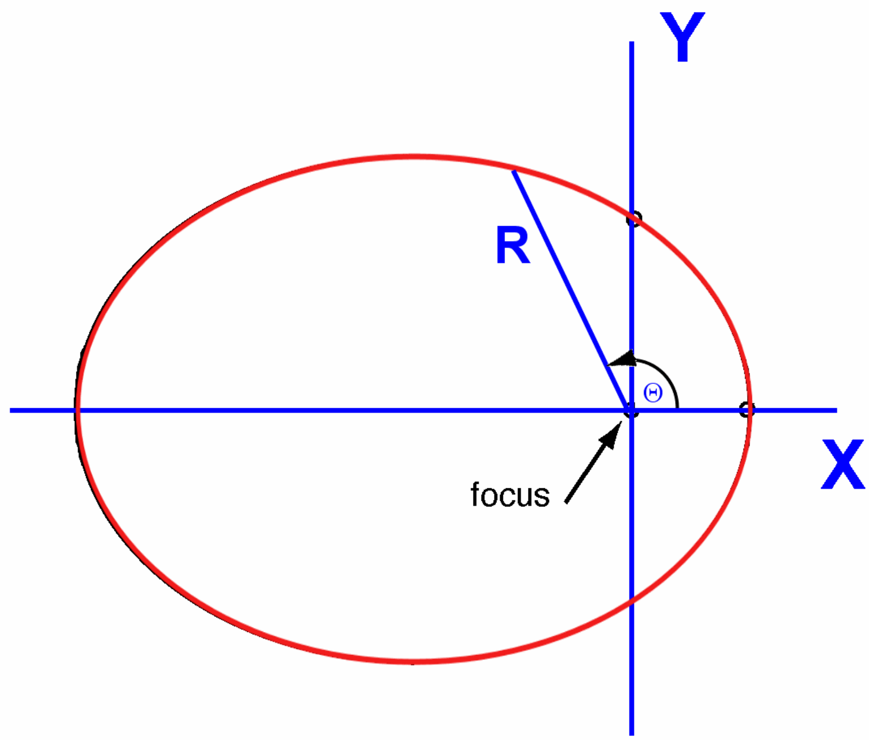


Figure 3

This angle, called E here, is called the Eccentric Anomaly in satellite work and the Reduced Latitude in geodesy. (It is usually denoted by β when called the reduced latitude.) A capital letter is used for the eccentric anomaly to match the common usage.

This angle is determined by drawing a line parallel to the y-axis through the point of interest on the ellipse. A circle is drawn around the ellipse with radius, a, the semi-major axis. The line from the center of the ellipse to the intersection of this vertical line and this circle defines the angle E, the eccentric anomaly. (The definition of E can also be formulated in terms of lines parallel to the x-axis and intersecting a circle of radius b, the semi-minor axis.) This auxiliary angle is used quite often in satellite work. It is also used in some geodetic formulae.

The last angle is used almost exclusively in satellite and astronomic work. This is the True Anomaly, Θ.



Focus Centered Coordinates
True Anomaly and R

Figure 4

This is the angle measured from the X-axis in the focus centered coordinated system.

III. Two and Three Dimensional Cases

In the equations that follow, the two dimensional notation is used. This may be confusing at times because the geodesy terminology of latitude is also used. The following changes need to be made to move from the two dimensional x,y notation here and the three dimensional x,y,z case of a solid figure.

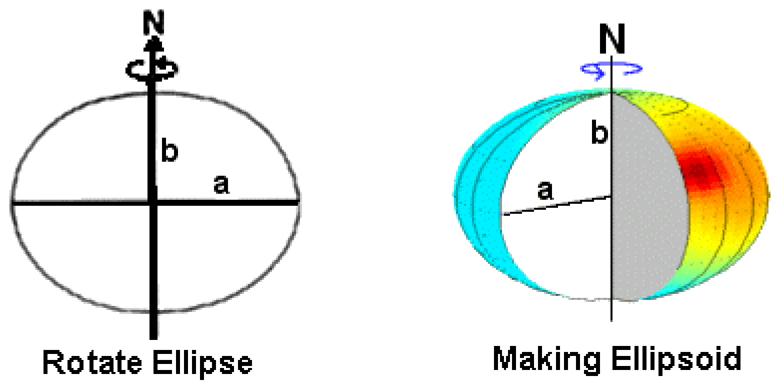


Figure 5

Two Dimensional	Three Dimensional
x-axis	distance from the polar axis
y-axis	axis of rotation, z-axis

IV. Important Notes

A. Equations Valid Only on Ellipse

The equations that follow are valid only on an ellipse.

They are not valid off the ellipse. In geodesy any point not on the ellipsoid is not on the ellipse as far as these equations are concerned. See the other notes in the geodesy section of clynchg3c.com for details. In particular look in the Technical Notes section at:

> "Geodetic Coordinate Conversions", Clynch, J.R., Feb 2006 (coordcvt.pdf)
>
> "Radius of the Earth – Radii Used in Geodesy", Clynch, J.R., Feb 2006 (radiigeo.pdf)

B. Spherical Polar Coordinates

None of the sets of angles and radii is precisely the spherical polar coordinates used in physics. The radius from the center, r is the same. The longitude is the same as the angle about the z-axis. However the angle of spherical polar coordinates measured form the z-axis is not here. This angle is the co-geocentric-latitude.

There are several physical quantities that are expressed as coefficients in a “spherical-polar like” system. The gravity potential as well as the magnetic field of the earth are represented this way for precision work. They usually expressly use the associated Legendre polynomials rather than the spherical harmonics. They use the geocentric latitude. This causes the sine and cosine functions to be exchanged from the form found in physics books. (Also carefully note the normalization of the coefficients. Three types are common, full, geophysics, and Schmidt.)

V. Common Notations.

Table 1
Ellipse Terminology

Symbol	Parameter	Other symbols
A	Point of apogee	
a	semi-major axis	
b	semi-minor axis	
c	half focal separation	ae, ε
E	Eccentric, Parametric, or reduced angle or eccentric anomaly, reduced latitude (β)	β,t, e, υ,
e	(first) eccentricity	ε
e'	Second eccentricity	ε'
F	foci	
f	flattening or first flattening (also sometimes elliptically)	
f'	Second flattening	
M	Mean anomaly	
P	Point of perigee	
P(x, y)	Points on the ellipse	Q, many
p	Semi-latus rectum	
R	Radial distance from focus	r
R_M	Radius of curvature in meridian direction	M
R_N	radius of curvature in prime vertical	N, ν, R_ν
r	radial distance from center	
S	Distance from focus to ellipse	
α	angular eccentricity	
ε	Linear eccentricity	E
Θ	True anomaly	f, θ, v, ψ
φ'	central or geocentric angle	θ
φ	Geodetic latitude	

VI. Ellipse Equations

Table 2
Cartesian Components and Ratios

Cartesian	Geocentric (centric) ϕ' or θ	Eccentric E	Geodetic ϕ
$x =$	$r\cos\phi'$	$a\cos E$	$R_N\cos\phi$ $\frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}$ $\frac{a^2\cos\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}$
$y =$	$r\sin\phi'$	$b\sin E$	$R_N\frac{b^2}{a^2}\sin\phi$ $R_N(1-e^2)\sin\phi$ $\frac{b}{a}\frac{b\sin\phi}{\sqrt{1-e^2\sin^2\phi}}$ $\frac{b^2\sin\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}$
$\frac{y}{x} =$	$\tan\phi'$	$\frac{b}{a}\tan E$	$\frac{b^2}{a^2}\tan\phi$ $(1-e^2)\tan\phi$
The Above Relations in the tangents are often used to convert angles			

Table 3

Radial distance from the ellipse center for the three center-origin angles.

Cartesian	Geocentric	Eccentric	Geodetic
$r^2 =$ $\sqrt{x^2 + y^2} =$	$\dfrac{a^2}{1+e'^2 \sin^2 \phi'}$	$a^2(1-e^2 \sin^2 E)$	$R_N^2[\cos^2 \phi + \dfrac{b^4}{a^4} \sin^2 \phi]$
	$\dfrac{b^2}{1-e^2 \cos^2 \phi'}$ $\dfrac{a^2(1-e^2)}{1-e^2 \cos^2 \phi'}$	$a^2 \cos^2 E + b^2 \sin^2 E$	$\dfrac{a^4 \cos^2 \phi + b^4 \sin^2 \phi}{a^2 \cos^2 \phi + b^2 \sin^2 \phi}$
	$= \dfrac{a^2 b^2}{a^2 \sin^2 \phi' + b^2 \cos^2 \phi'}$		

Table 4

Center origin angle conversions

Cartesian	Eccentric	Geodetic	Geodetic-R_N
$\frac{x}{a} =$	$\cos E$	$= \frac{\cos\phi}{\sqrt{1-e^2\sin^2\phi}}$	$= \frac{R_N}{a}\cos\phi$
$\frac{y}{b} =$	$\sin E$	$= \frac{b}{a}\frac{\sin\phi}{\sqrt{1-e^2\sin^2\phi}}$	$= \frac{R_N b}{a^2}\sin\phi$
Cartesian	Geocentric	Geodetic	Geodetic-R_N
$\frac{x}{r} =$	$\cos\phi'$	$= \frac{a}{r}\frac{\cos\phi}{\sqrt{1-e^2\sin^2\phi}}$	$= \frac{R_N}{r}\cos\phi$
$\frac{y}{r} =$	$\sin\phi'$	$= \frac{b^2}{a\,r}\frac{\sin\phi}{\sqrt{1-e^2\sin^2\phi}}$	$= \frac{R_N b^2}{r\,a^2}\sin\phi$ $= \frac{R_N}{r}(1-e^2)\sin\phi$
Cartesian	Geocentric	Eccentric	
$x =$	$r\cos\phi'$	$a\cos E$	
$y =$	$r\sin\phi'$	$b\sin E$	
$\frac{x}{r} =$	$\cos\phi'$	$\frac{a}{r}\cos E$	
$\frac{y}{r} =$	$\sin\phi'$	$\frac{b}{r}\sin E$	

Table 5

Conversion between eccentric anomaly and true anomaly.

Origin at Focus with Minimum R at $\Theta = 0$

Cartesian	True Anomaly	Eccentric
$X = x - c =$ $x - ae =$	$R\cos\Theta$	$a(\cos E - e)$
$Y = y =$	$R\sin\Theta$	$b\sin E$
$\frac{X}{R} =$	$\cos\Theta$	$\frac{\cos E - e}{1 - e\cos E}$ $\frac{a\cos E - ae}{a(1 - e\cos E)}$
$\frac{Y}{R} =$	$\sin\Theta$	$\frac{\sqrt{1-e^2}}{1 - e\cos E}\sin E$ $\frac{b}{a}\frac{\sin E}{(1 - e\cos E)}$
$R =$	$= \frac{a(1-e^2)}{1+e\cos\Theta}$ $= \frac{b^2}{a}\frac{1}{1+e\cos\Theta}$	$a(1 - e\cos E)$
	$\frac{\cos\Theta + e}{1 + e\cos\Theta}$	$= \cos E$
	$\frac{R}{b}\sin\Theta$	$= \sin E$

Table 6
Auxiliary Quantities and Relations

R_N	$= \dfrac{a}{\sqrt{1-e^2 \sin^2 \phi}}$ $= \dfrac{a^2}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}$
R_M	$= \dfrac{a\,(1-e^2)}{(1-e^2 \sin^2 \phi)^{3/2}}$ $= R_N \ \dfrac{1-e^2}{1-e^2 \sin^2 \phi}$
e^2	$= \dfrac{a^2 - b^2}{a^2}$ $= 1 \ - \ \dfrac{b^2}{a^2}$
$1-e^2$	$= \dfrac{b^2}{a^2}$
e'^2	$= \dfrac{a^2 - b^2}{b^2}$ $= \dfrac{a^2}{b^2} \ - \ 1$
$\dfrac{e}{e'}$	$= \dfrac{b}{a}$
Foci location (+/- c)	$c \ = ae \ = \sqrt{a^2 - b^2}$
Mean Anomaly M	$2\pi \, t/T$ T = period of orbit, t = time since Rmin
Kepler's Equation	$M \ = E - e \sin E$
True Anomaly to Eccentric	$\tan \dfrac{\Theta}{2} \ = \ \sqrt{\dfrac{1+e}{1-e}} \ \tan \dfrac{E}{2}$