Summary of Vector Properties

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Basics Quantities - Scalars and Vectors

There are quantities, such as velocity, that have both magnitude and direction. These are vectors. A quantity with only magnitude is a scalar. It can be positive or negative.

Nomenclature

$\vec{A}, \vec{B}, \vec{C}, \vec{D}$	will be vectors
$\lvert\vec{A}\rvert$	is the magnitude or length of the vector $\vec{A}$. It is a positive scalar.
$\hat{e}_x, \hat{e}_y, \hat{e}_z$	will be a set of orthogonal unit vectors along the x,y,z axes
	(Some authors use **i,j,k** for these unit vectors.)
A_x, A_y, A_z	are scalars that are the components of the vector $\vec{A}$
s	is a scalar

Component Notations

$$\begin{aligned}\vec{A} &= A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z \\ &= (A_x, A_y, A_z)\end{aligned}$$

Components can be thought of as projections (inner products) on the basis vectors.

$$\begin{aligned} A_x &= \vec{A} \bullet \hat{e}_x = \hat{e}_x \bullet \vec{A} \\ A_y &= \vec{A} \bullet \hat{e}_y = \hat{e}_y \bullet \vec{A} \\ A_z &= \vec{A} \bullet \hat{e}_z = \hat{e}_z \bullet \vec{A} \end{aligned}$$

Scalar Multiplication

$$s\,\vec{A} = s\,(A_x, A_y, A_z) = (sA_x, sA_y, sA_z)$$

$$\begin{aligned} s\,(\vec{A} + \vec{B}) &= s\,\vec{A} + s\,\vec{B} \\ &= (sA_x + sB_x, sA_y + sB_y, sA_z + sB_z) \end{aligned}$$

Addition

$$\begin{aligned} \vec{A} + \vec{B} &= \vec{B} + \vec{A} \\ &= (A_x + B_x, A_y + B_y, A_z + B_z) \end{aligned}$$

Dot Product or Inner Product

The Dot product of $\vec{A}$ and $\vec{B}$, denoted as $\vec{\mathrm{A}} \bullet \vec{\mathrm{B}}$, is a scalar.

$$\vec{\mathrm{A}} \bullet \vec{\mathrm{B}} \;=\; \left|\vec{\mathrm{A}}\right| \left|\vec{\mathrm{B}}\right| \cos\theta$$

$$= \; \mathrm{A_x B_x + A_y B_y + A_z B_z}$$

Where θ is the angle between $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$. (Either angle can be used.)

$$\vec{\mathrm{A}} \bullet \vec{\mathrm{B}} \;=\; \vec{\mathrm{B}} \bullet \vec{\mathrm{A}}$$

$$\vec{\mathrm{A}} \bullet (\vec{\mathrm{B}} + \vec{\mathrm{C}}) \;=\; \vec{\mathrm{A}} \bullet \vec{\mathrm{B}} + \vec{\mathrm{A}} \bullet \vec{\mathrm{C}} \;=\; (\vec{\mathrm{B}} + \vec{\mathrm{C}}) \bullet \vec{\mathrm{A}}$$

$$\mathrm{s}(\vec{\mathrm{A}} \bullet \vec{\mathrm{B}}) \;=\; (\mathrm{s}\vec{\mathrm{A}}) \bullet \vec{\mathrm{B}} \;=\; \vec{\mathrm{A}} \bullet (\mathrm{s}\vec{\mathrm{B}})$$

Magnitude or Length

$$\left|\vec{\mathrm{A}}\right| \;=\; \sqrt{\vec{\mathrm{A}} \bullet \vec{\mathrm{A}}}$$

$$= \; \sqrt{\mathrm{A_x^2 + A_y^2 + A_z^2}}$$

Cross Product or Outer Product

$\vec{\mathrm{D}} \;=\; \vec{\mathrm{A}} \times \vec{\mathrm{B}}$ is a vector (pseudo-vector). It is perpendicular to both $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ which implies that: $0 \;=\; \vec{\mathrm{D}} \bullet \vec{\mathrm{A}} \;=\; \vec{\mathrm{D}} \bullet \vec{\mathrm{B}}$

$$\left|\vec{\mathrm{D}}\right| = \left|\mathrm{A}\right| \left|\mathrm{B}\right| \sin\theta$$

The direction of $\vec{\mathrm{D}}$ is perpendicular to $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$, in the direction of a right hand screw advance when $\vec{\mathrm{A}}$ is rotated into $\vec{\mathrm{B}}$. (Right hand rule.)

The cross product is not commutative,

$$\vec{\mathrm{A}} \times \vec{\mathrm{B}} \;=\; -\,\vec{\mathrm{B}} \times \vec{\mathrm{A}}$$

$$\vec{\mathrm{D}} \;=\; \vec{\mathrm{A}} \times \vec{\mathrm{B}}$$

$$= \; (\; \mathrm{A_y B_z - A_z B_y}, \quad \mathrm{A_y B_z - A_z B_y}, \quad \mathrm{A_y B_z - A_z B_y} \;)$$

Note the cyclic nature of the multiplications in component notation.

Formally, manipulating symbols, we can represent the cross product as a determinate.

$$
\begin{aligned}
\vec{D} &= \vec{A}\times\vec{B} \\
&= \det\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
\end{aligned}
$$

Orthogonal Unit (Orthonormal) Vectors

From the properties of the inner product one has:

$$
\begin{aligned}
\hat{e}_x \bullet \hat{e}_x &= \hat{e}_y \bullet \hat{e}_y = \hat{e}_z \bullet \hat{e}_z = 1 \\
\hat{e}_x \bullet \hat{e}_y &= \hat{e}_x \bullet \hat{e}_z = \hat{e}_y \bullet \hat{e}_z = 0
\end{aligned}
$$

The cross products are:

$$
\begin{aligned}
\hat{e}_x \times \hat{e}_y &= \hat{e}_z = -(\hat{e}_y \times \hat{e}_x) \\
\hat{e}_y \times \hat{e}_z &= \hat{e}_x = -(\hat{e}_z \times \hat{e}_y) \\
\hat{e}_z \times \hat{e}_x &= \hat{e}_y = -(\hat{e}_x \times \hat{e}_z)
\end{aligned}
$$

The magnitude of unit vectors cross products are:

$$
\begin{aligned}
|\hat{e}_x \times \hat{e}_y| &= |\hat{e}_y \times \hat{e}_z| = |\hat{e}_z \times \hat{e}_x| = 1 \\
|\hat{e}_x \times \hat{e}_x| &= |\hat{e}_y \times \hat{e}_y| = |\hat{e}_z \times \hat{e}_z| = 0
\end{aligned}
$$

Vectors and Pseudovectors

If you invert all three axis of the coordinate system you change from a right handed system to a left handed system. In this case a vector stays the same, although its component are the negative of the previous values. (The components times the unit vectors are invariant.) A pseudovector will change sign under this operation. In this sense, a cross product is a pseudovector. In most physics and engineering this distinction has no significance.